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CARDINAL INTERPOLATING MULTIRESOLUTIONS *

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Abstract. The basic method of constructing wavelets is by means of a multiresolution approximation of $\mathbf{L}^2(\mathbf{R})$. In this paper we present a class of multiresolution approximations for which the associated scaling function has a simple cardinal interpolation property. We present the construction of such multiresolutions and discuss the symmetry, decay, and regularity properties of the associated scaling functions and wavelets.

Key Words. Wavelets, interpolation.

1. Introduction. Wavelets have recently received a great deal of attention in such areas as signal processing, image processing, and the numerical computation of singular integral operators. In its simplest form, a wavelet is a function ψ such that the family

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \qquad j, k \in \mathbf{Z}$$

forms an orthonormal basis for $\mathbf{L}^2(\mathbf{R})$. Examples of such ψ have been constructed for which the $\psi_{j,k}$ also form bases for such function spaces as the Sobolev spaces $H^s(\mathbf{R})$. Moreover, wavelets naturally exhibit a marked degree of spatial and frequency localization, which accounts for much of their utility.

A simple and elegant way both to construct wavelet bases and to compute the wavelet expansion of a function is by means of a *multiresolution approximation* ([1], [6], [8]). A multiresolution approximation $\mathbf{M}$ of $\mathbf{L}^2(\mathbf{R})$ is a sequence $(\mathbf{V}_j)$, $j \in \mathbf{Z}$ of closed subspaces of $\mathbf{L}^2(\mathbf{R})$ such that
1. $\mathbf{V}_j \subset \mathbf{V}_{j+1}$,
2. $\bigcup_{j=-\infty}^{\infty} \mathbf{V}_j$ is dense in $\mathbf{L}^2(\mathbf{R})$ and $\bigcap_{j=-\infty}^{\infty} \mathbf{V}_j = 0$,
3. $f(x) \in \mathbf{V}_j \Longleftrightarrow f(2x) \in \mathbf{V}_{j+1}$,
4. $f(x) \in \mathbf{V}_j \Longleftrightarrow f(x - 2^{-j}k) \in \mathbf{V}_j$, and
5. There exists a function $g \in \mathbf{V}_0$ such that the family $(g(x-k))_{k \in \mathbf{Z}}$ forms a Riesz basis for $\mathbf{V}_0$.

Of paramount importance is the following consequence of condition (5):
5′. There exists a function ϕ such that the family $(\phi(x-k))_{k \in \mathbf{Z}}$ forms an orthonormal basis for $\mathbf{V}_0$.

The function ϕ is called the *scaling function* associated with the multiresolution. In this notation the index j indicates scale while the index k indicates spatial location.

The scaling and translation properties (3) and (4) imply that the family

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \qquad j, k \in \mathbf{Z}$$

forms an orthonormal basis for $\mathbf{V}_j$. From ϕ we construct the wavelet ψ.

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In this paper we will study multiresolution approximations of $\mathbf{L}^2(\mathbf{R})$ for which the associated scaling functions Φ are continuous and have the following cardinal interpolation property: For $k \in \mathbf{Z}$,

$$\Phi(k) = \delta(k) \equiv \left\{ \begin{array}{ll} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{array} \right.$$

We will call a multiresolution for which the scaling function Φ is continuous and satisfies this interpolation property a *cardinal interpolating multiresolution*. Such multiresolutions are also discussed in [5].

The main result of this paper is the construction of examples of interpolating multiresolutions. We will give a simple condition which will allow us to construct examples of cardinal interpolating multiresolutions and we will study some of their properties. We will also show how to construct cardinal interpolating multiresolutions from other multiresolutions. We will construct examples of cardinal interpolating multiresolutions from the spline multiresolutions of Battle and Lemarie [4] and from the compactly supported multiresolutions of Daubechies [2].

These constructions will produce families of cardinal interpolating multiresolutions for which the scaling functions Φ and wavelets Ψ are real-valued, exponentially decaying functions which can be designed to have an arbitrarily high degree of smoothness and an arbitrarily large number of vanishing moments. As we will discuss in Section 9, such interpolating multiresolutions provide an attractive and natural way of computing from sampled data the initial projection needed for Mallat's cascade wavelet transform algorithm [7].

We begin in Section 2 with a brief review of some properties of multiresolutions that we will need. In Section 3 we present the condition which allows us to construct cardinal interpolating multiresolutions. In Sections 4—7 we discuss some of the restrictions that the interpolation property places on the multiresolution. In Section 8 we show how the interpolation condition leads to some interesting relations involving the values of the scaling function Φ and the wavelet Ψ. In Section 9 we discuss the vanishing moments properties of interpolating multiresolutions.

The main results of this paper are contained in Sections 11 – 13. There we show how an arbitrary multiresolution can be used to construct a cardinal interpolating multiresolution and we investigate the properties of the cardinal interpolating multiresolutions so constructed. In Sections 12 and 13 we present two families of cardinal interpolating multiresolutions.

To prevent any confusion over normalization, the definition of the Fourier transform we use in this paper is

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dx \, f(x) e^{-i\omega x}.$$

Its inverse is then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \hat{f}(\omega) e^{i\omega x}.$$

For a generic multiresolution we will use the symbols ϕ and ψ for the associated scaling function and wavelet. We will reserve Φ and Ψ for the scaling function and wavelet associated with a cardinal interpolating multiresolution.

2. Multiresolution approximations. In this section we will review those features of *multiresolution approximations* (or multiresolutions, for short) which we will need. For a fuller discussion of this subject, we refer the reader to [1], [6], and [8].

Following A. Cohen ([1]), we will say that a multiresolution is *regular* if the scaling function $\phi(x)$ decays so quickly that $(1 + |x|)^n \phi(x) \in \mathbf{L}^2(\mathbf{R})$ for all $n \geq 0$. If a multiresolution is regular, then its properties are contained in what we will call the *scale-transition filter* $H(\omega)$. This is a function which describes how to pass between two levels $\mathbf{V}_j$ and $\mathbf{V}_{j+1}$ of the multiresolution.

Since $\phi(\frac{x}{2}) \in \mathbf{V}_{-1} \subset \mathbf{V}_0$, we have

$$\frac{1}{2}\phi\left(\frac{x}{2}\right) = \sum_{k=-\infty}^{\infty} h_k \phi(x - k),$$

where

$$(1) \qquad h_k = \frac{1}{2} \int dx \, \phi\left(\frac{x}{2}\right) \phi(x - k).$$

We define the scale-transition filter $H(\omega)$ to be

$$H(\omega) = \sum_{k=-\infty}^{\infty} h_k e^{ik\omega};$$

then

$$(2) \qquad \hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega).$$

$H(\omega)$ is $\mathbf{C}^{\infty}$ because of the decay rate of ϕ. Since $(\phi(x - k))_{k \in \mathbf{Z}}$ forms an orthonormal family, $H(\omega)$ satisfies

$$(3) \qquad H(0) = 1,$$

$$(4) \qquad |H(\omega)|^2 + |H(\omega + \pi)|^2 = 1.$$

As a consequence of (2) we have the fundamental relation

$$(5) \qquad \hat{\phi}(\omega) = \prod_{k=1}^{\infty} H(2^{-k}\omega).$$

From (3) and (5) we see that ϕ has unit mean:

$$(6) \qquad \hat{\phi}(0) = 1.$$

In order to define the wavelet ψ, we first define the filter $G(\omega)$ by

$$(7) \qquad G(\omega) = e^{-i\omega}\overline{H(\omega + \pi)}.$$

The wavelet ψ is then defined via

$$(8) \qquad \hat{\psi}(\omega) = G\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right).$$

3. The interpolation condition. We define an *interpolating multiresolution* to be a regular multiresolution for which the scaling function Φ is real-valued, continuous, and satisfies

$$(9) \qquad\qquad \Phi(k) = \delta(k).$$

We should remark that the structure of multiresolutions places *a priori* restrictions on the kinds of interpolation properties we might consider. One property of multiresolutions is that the scaling function Φ necessarily satisfies

$$\sum_{k=-\infty}^{\infty} \Phi(x-k) = 1.$$

(See [8].) In particular, we must have

$$\sum_{k=-\infty}^{\infty} \Phi(k) = 1.$$

This property limits the interpolation condition we might hope to impose on the scaling function. In view of this restriction the condition (9) is not only the simplest interpolation property but the most natural for multiresolutions.

We will derive conditions on the scale-transition filter $H(\omega)$ under which the associated scaling function Φ has the interpolation property (9). We begin with the following well-known formula [8]: Suppose $f, \hat{f} \in \mathbf{L}^1$, and let $F(\omega) = \sum_{k=-\infty}^{\infty} \hat{f}(\omega + 2k\pi)$. Then $F \in \mathbf{L}^1[0, 2\pi]$ and

$$F(\omega) = \sum_{k=-\infty}^{\infty} \hat{f}(\omega + 2k\pi) = \sum_{k=-\infty}^{\infty} f(k)e^{ik\omega}.$$

In the case of an interpolating multiresolution the preceding relation reduces to

$$\sum_{k=-\infty}^{\infty} \hat{\Phi}(\omega + 2k\pi) = 1.$$

Now suppose that $H(\omega)$ is the scale-transition filter associated with the multiresolution. Since $\hat{\Phi}(2\omega) = H(\omega)\hat{\Phi}(\omega)$,

$$\sum_{k=-\infty}^{\infty} \hat{\Phi}(2\omega + 2k\pi) = \sum_{k=-\infty}^{\infty} H(\omega + k\pi)\hat{\Phi}(\omega + k\pi) = 1.$$

Now break up the sum over even and odd k. Since H is 2π-periodic,

$$H(\omega) \sum_{k=-\infty}^{\infty} \hat{\Phi}(\omega + 2k\pi) + H(\omega + \pi) \sum_{k=-\infty}^{\infty} \hat{\Phi}(\omega + \pi + 2k\pi) = H(\omega) + H(\omega + \pi).$$

Thus H must satisfy the relation

$$(10) \qquad\qquad H(\omega) + H(\omega + \pi) = 1.$$

The condition (10) is sufficient to generate an interpolating multiresolution under certain other assumptions on $H(\omega)$. This is the import of the following theorem, which is a simple elaboration on the sufficient conditions concerning the construction of a multiresolution from a scale-transition filter $H(\omega)$ in [1] and [6].

THEOREM 3.1. *Suppose $H(\omega)$ has the following properties:*

(11) $$H(\omega) \neq 0 \text{ for all } \omega \in [-\pi/2, \pi/2],$$

(12) $$H(\omega) \text{ is } \mathbf{C}^\infty \text{ and } 2\pi\text{-periodic},$$

(13) $$H(0) = 1,$$

(14) $$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 1.$$

Define

$$\hat{\Phi}(\omega) = \prod_{k=1}^{\infty} H(2^{-k}\omega).$$

A. *Then Φ defines a regular multiresolution of $\mathbf{L}^2(\mathbf{R})$. The function $\hat{\Phi}(\omega)$ is the Fourier transform of a function $\Phi(x)$ such that $(\Phi(x-k))_{k \in \mathbf{Z}}$ is an orthonormal basis for a closed subspace $\mathbf{V}_0$ of $L^2(\mathbf{R})$, and $(1 + |x|)^n \Phi(x) \in \mathbf{L}^2(\mathbf{R})$ for all $n \geq 0$.*

B. *If, in addition, $\hat{\Phi}$ is integrable and $H(\omega)$ satisfies*

$$H(\omega) + H(\omega + \pi) = 1,$$

then Φ is continuous and defines a cardinal interpolating multiresolution.

We will need the following lemma for the proof of this theorem. This lemma is a slightly generalized restatement of Lemma 2 in [6].

LEMMA 3.2. *Suppose $m(\omega)$ is a continuous 2π-periodic function satisfying*

$$m(\omega) + m(\omega + \pi) = 1.$$

For $k \geq 1$ define $m_k(\omega)$ to be

$$m_k(\omega) = \begin{cases} m\left(\frac{\omega}{2}\right) m\left(\frac{\omega}{4}\right) \cdots m\left(\frac{\omega}{2^k}\right), & \text{if } |\omega| \leq 2^k \pi, \\ 0, & \text{if } |\omega| > 2^k \pi. \end{cases}$$

Then for all $k \geq 1$ we have

$$I_k^n \equiv \int_{-\infty}^{\infty} d\omega \, m_k(\omega) e^{in\omega} = \begin{cases} 2\pi, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Part **A** of Theorem 3.1 is contained in [1] and [6]. The proof of part **B** uses the techniques from the proof of part **A**.

Proof. We must show that $\Phi(k) = \delta(k)$. Since $\hat{\Phi} \in \mathbf{L}^1$, Φ is continuous, and it makes sense to speak of the pointwise values $\Phi(k)$.

Define

$$\hat{\Phi}_k(\omega) = \begin{cases} H\left(\frac{\omega}{2}\right) H\left(\frac{\omega}{4}\right) \cdots H\left(\frac{\omega}{2^k}\right), & \text{if } |\omega| \leq 2^k \pi, \\ 0, & \text{if } |\omega| > 2^k \pi. \end{cases}$$

Applying Lemma 3.2 we see that

$$\int_{-\infty}^{\infty} d\omega \, \hat{\Phi}_k(\omega) e^{in\omega} = 2\pi \delta(n).$$

We now will apply the dominated convergence theorem to conclude the same about $\hat{\Phi}$.

The conditions (12), (13), and (11) imply there exists a lower bound $c > 0$ such that for all $\omega \in [-\pi, \pi]$,

$$\left| \hat{\Phi}(\omega) \right| \geq c.$$

(See [1] or [6] for a proof.) For $|\omega| \leq 2^k \pi$ we have

$$\hat{\Phi}(\omega) = \hat{\Phi}_k(\omega) \hat{\Phi}\left(\frac{\omega}{2^k}\right),$$

so for $|\omega| \leq 2^k \pi$,

$$(15) \qquad\qquad \left| \hat{\Phi}_k(\omega) \right| \leq \frac{1}{c} \left| \hat{\Phi}(\omega) \right|.$$

Since $\hat{\Phi}_k(\omega) = 0$ for $|\omega| > 2^k \pi$, (15) holds for all ω. Since we assumed that $\hat{\Phi}$ is integrable, we may apply the dominated convergence theorem to conclude that

$$\Phi(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \hat{\Phi}(\omega) e^{in\omega} = \delta(n).$$

□

4. Restrictions on the real and imaginary parts of H. In this section we will summarize the conditions we need to impose on the real and imaginary parts of the filter $H(\omega)$ in order to produce a cardinal interpolating multiresolution. These conditions will be used in Section 11, where we will present a large class of cardinal interpolating multiresolutions.

The filter $H(\omega)$ must satisfy the two conditions

$$(16) \qquad\qquad H(\omega) + H(\omega + \pi) = 1$$

$$(17) \qquad\qquad |H(\omega)|^2 + |H(\omega + \pi)|^2 = 1.$$

These yield $|H(\omega)|^2 - \mathrm{Re}\, H(\omega) = 0$, or

$$(18) \qquad \left| H(\omega) - \frac{1}{2} \right| = \frac{1}{2}.$$

Let $H(\omega) = a(\omega) + i\, b(\omega)$. Then (18) leads to the relation

$$(19) \qquad b^2(\omega) = a(\omega) - a^2(\omega).$$

In addition, in order to satisfy (16), $a(\omega)$ and $b(\omega)$ must also satisfy

$$(20) \qquad a(\omega) + a(\omega + \pi) = 1$$
$$(21) \qquad b(\omega) + b(\omega + \pi) = 0.$$

Finally, if Φ is real, then $H(\omega) = \overline{H(-\omega)}$, so a and b must satisfy

$$(22) \qquad a(\omega) = a(-\omega)$$
$$(23) \qquad b(\omega) = -b(-\omega).$$

Conversely, it is easy to see from the definition of $\hat{\Phi}$ that if a and b satisfy the two preceding relations then Φ will be real.

5. Missing Fourier coefficients. The scale-transition filter $H(\omega)$ associated with a cardinal interpolating multiresolution has the following curious property: All but one of the "even" Fourier coefficients of $H(\omega)$ vanish. If $H(\omega) = \sum_{k=-\infty}^{\infty} h_k e^{ik\omega}$, then

$$\sum_{k=-\infty}^{\infty} h_k e^{ik\omega} + \sum_{k=-\infty}^{\infty} h_k e^{ik(\omega+\pi)} = \sum_{k=-\infty}^{\infty} 2h_{2k} e^{i2k\omega} = 1.$$

Thus

$$(24) \qquad h_{2k} = \begin{cases} 1/2, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$$

In view of (1), this can be interpreted in terms of orthogonality relations between $\phi(x/2)$ and the family $(\phi(x - k))_{k \in \mathbf{Z}}$.

6. Asymmetry of Φ. The scaling functions associated with cardinal interpolating multiresolutions have an annoying lack of symmetry. In fact, the interpolating scaling function Φ cannot be symmetric with respect to the origin. Suppose that $\Phi(x)$ is symmetric with respect to the origin. Then the Fourier coefficients h_k of $H(\omega)$ are symmetric in k: $h_k = h_{-k}$. Accordingly, $H(\omega) = \sum_{k=-\infty}^{\infty} h_k e^{ik\omega}$ is purely real, so

$$(\mathrm{Im}\, H(\omega))^2 = a(\omega) - a^2(\omega) \equiv 0.$$

Since $a(\omega)$ is continuous we must have either $a \equiv 0$ or $a \equiv 1$. Since $H(0) = 1$, it follows that $a(\omega) \equiv 1$, which contradicts the condition $a(\omega) + a(\omega + \pi) = 1$.

7. Non-existence of compactly supported interpolating multiresolutions.
The condition (18) restricts the values of $H(\omega)$ to lie on the circle $\{\zeta \in \mathbf{C} : \left|\zeta - \frac{1}{2}\right| = \frac{1}{2}\}$.
As a consequence, the only compactly supported multiresolution with the interpolation
property $\Phi(k) = \delta(k)$ corresponds to the Haar basis, for which the scaling function is
not continuous.

To prove this, we first note the following fact: If $p(\zeta)$ is a polynomial and $|p(\zeta)| = 1$
for all $|\zeta| = 1$, then $p(\zeta) = c\zeta^n$ for some $n \geq 0$ and $|c| = 1$ (see, for instance, 65,
Part VI, of [9]). Now suppose that $H(\omega)$ has a finite Fourier expansion, as required
by Daubechies' characterization of the scale-transition filters associated with compactly
supported multiresolutions [2]. Then $H(\omega) = p(e^{i\omega})$ for some polynomial p, and from
(3), (18), the aforementioned fact about polynomials, and (24), we see that

$$H(\omega) - \frac{1}{2} = \frac{1}{2}e^{in\omega}$$

for some odd n. The only such choice of n that leads to a multiresolution is

$$H(\omega) = \frac{1}{2} + \frac{1}{2}e^{i\omega},$$

which generates the Haar basis (see [1]).

8. Values of Φ and Ψ at dyadic points. For cardinal interpolating multiresolutions there exist simple identities for the values of $\Phi(k2^{-j})$ and $\Psi(k2^{-j})$ for any
$j, k \in \mathbf{Z}$.

Since $\hat{\Phi}(2\omega) = H(\omega)\hat{\Phi}(\omega)$, for $j \geq 1$ we have

$$\hat{\Phi}(2^j\omega) = H(2^{j-1}\omega)\hat{\Phi}(2^{j-1}\omega) = \ldots = \left[\prod_{k=0}^{j-1} H(2^k\omega)\right]\hat{\Phi}(\omega).$$

Let $\Phi_j(x) = \Phi(2^{-j}x)$. For $j \geq 1$ we have

$$\hat{\Phi}_j(\omega) = 2^j\hat{\Phi}(2^j\omega) = 2^j\left[\prod_{k=0}^{j-1} H(2^k\omega)\right]\hat{\Phi}(\omega).$$

Since H is 2π-periodic,

$$\sum_{k=-\infty}^{\infty} \hat{\Phi}_j(\omega + 2k\pi) = 2^j\left[\prod_{k=0}^{j-1} H(2^k\omega)\right]\sum_{k=-\infty}^{\infty} \hat{\Phi}(\omega + 2k\pi).$$

Because Φ is associated with a cardinal interpolating multiresolution, this expression reduces to a particularly simple form. Since Φ satisfies (9), we have

$$\sum_{k=-\infty}^{\infty} \Phi(k2^{-j})e^{ik\omega} = \sum_{k=-\infty}^{\infty} \hat{\Phi}_j(\omega + 2k\pi) = 2^j\prod_{k=0}^{j-1} H(2^k\omega).$$

From this relation the values $\Phi(k2^{-j})$ can be computed in terms of the Fourier coefficients of a finite product. This identity has another interpretation. The substitution $\omega \leftarrow k2^{-j}\omega$ yields

$$(25) \qquad 2^{-j} \sum_{k=-\infty}^{\infty} \Phi(k2^{-j})e^{ik2^{-j}\omega} = \prod_{k=1}^{j-1} H(2^{-k}\omega).$$

On the left-hand side we have a rectangular rule approximation to $\hat{\Phi}(\omega)$, while on the other side we have a truncated product approximation to $\hat{\Phi}(\omega)$.

Similarly, from (8) one obtains the identity

$$\sum_{k=-\infty}^{\infty} \Psi(k2^{-j})e^{ik\omega} = 2^j G(2^{j-1}\omega) \prod_{k=0}^{j-2} H(2^k\omega), \qquad j \geq 2$$

for the associated wavelet Ψ, from which one can compute the values $\Psi(k2^{-j})$ and derive a formula similar to (25).

9. Vanishing moments. In Sections 12 and 13 we will show how to construct cardinal interpolating multiresolutions for which the scaling functions Φ can have arbitrarily high numbers of vanishing moments. In this section we relate the property of vanishing moments to the scale-transition filter $H(\omega)$.

For an arbitrary multiresolution the number of vanishing moments of the scaling function ϕ and the wavelet ψ is related to the flatness of the scale-transition filter $H(\omega)$ at $\omega = 0$. First note that if $H(\omega) \in \mathbf{C}^{\infty}$, then $\hat{\phi}(\omega) \in \mathbf{C}^{\infty}$ and $\hat{\psi}(\omega) \in \mathbf{C}^{\infty}$. Moreover, from (2) and (8) we see that

$$(26) \qquad \frac{d^k H}{d\omega^k}(0) = 0 \text{ for } 1 \leq k \leq N \iff \frac{d^k \hat{\phi}}{d\omega^k}(0) = 0 \text{ for } 1 \leq k \leq N, \text{ and}$$

$$(27) \qquad \frac{d^k H}{d\omega^k}(\pi) = 0 \text{ for } 0 \leq k \leq N-1 \iff \frac{d^k \hat{\psi}}{d\omega^k}(0) = 0 \text{ for } 0 \leq k \leq N-1.$$

Recall also that $H((2k+1)\pi) = 0$ for all $k \in \mathbf{Z}$ because of (3) and (4), so we can write $H(\omega)$ in the form

$$H(\omega) = \cos^N \frac{\omega}{2} m(\omega),$$

for some $N \in \mathbf{Z}$ and some function $m(\omega)$.

In the case of a cardinal interpolating multiresolution, $H(\omega)$ also satisfies the interpolation condition (16):

$$H(\omega) + H(\omega + \pi) = 1.$$

Consequently, for $N \geq 1$ we have

$$(28) \qquad \frac{d^k H}{d\omega^k}(0) = 0 \text{ for } 1 \leq k \leq N \iff \frac{d^k H}{d\omega^k}(\pi) = 0 \text{ for } 1 \leq k \leq N.$$

This leads to the following proposition.

PROPOSITION 9.1. *Suppose that* $H(\omega)$ *is the scale-transition filter for a regular cardinal interpolating multiresolution* **I** *and suppose that*

$$H(\omega) = \cos^N \frac{\omega}{2} m(\omega),$$

where $m(\omega) \in \mathbf{C}^\infty$.
  Then

$$\int dx \, x^k \Phi(x) = 0 \quad \textit{for } 1 \le k \le N - 1$$

$$\int dx \, x^k \Psi(x) = 0 \quad \textit{for } 0 \le k \le N - 1.$$

Proof. The moment integrals make sense because of the rapid decay of Φ and Ψ. The presence of the factor $\cos^N(\omega/2)$ means that

$$\frac{d^k H}{d\omega^k}(\pi) = 0 \text{ for } 0 \le k \le N - 1.$$

The proposition now follows from (26), (27) and (28). $\square$

One useful consequence of the vanishing moments of Φ is the following error estimate. Suppose that $f \in \mathbf{L}^2(\mathbf{R}) \cap \mathbf{C}^N(\mathbf{R})$ and that $f^{(N)}(x)$ is bounded, and that we have a cardinal interpolating multiresolution for which the scaling function Φ is exponentially decaying and has at least $N - 1$ vanishing moments in the sense of Proposition 9.1. Given $\Delta x > 0$, rescale Φ by defining

$$\Phi_{\Delta x}(x) = (\Delta x)^{-\frac{1}{2}} \Phi\left(\frac{x}{\Delta x}\right).$$

The family $\Phi_{\Delta x}(x - k\Delta x)$, $k \in \mathbf{Z}$, is an orthonormal basis for a subspace $\mathbf{V}_0^{\Delta x}$ of $\mathbf{L}^2(\mathbf{R})$; this family of functions can be used to define a multiresolution of $\mathbf{L}^2$ where translation by k is replaced by translation by $k\Delta x$.

Now compute the coefficients d_k of the projection of f into $\mathbf{V}_0^{\Delta x}$, the subspace spanned by the $\Phi_{\Delta x}(x - k\Delta x)$, $k \in \mathbf{Z}$. Because of the vanishing moment properties of Φ, we have

$$d_k = \int dx \, f(x)\Phi_{\Delta x}(x - k\Delta x)$$

$$= \int dx \left[f(k\Delta x) + \sum_{n=1}^{N-1} \frac{f^{(n)}(k\Delta x)}{n!}(x - k\Delta x)^n + R^{(N)}(x) \right] \Phi_{\Delta x}(x - k\Delta x)$$

$$= \int dx \left[f(k\Delta x) + R^{(N)}(x) \right] \Phi_{\Delta x}(x - k\Delta x).$$

Recall from (6) that the scaling function Φ has unit mean. Thus

$$\int dx \, f(k\Delta x)\Phi_{\Delta x}(x - k\Delta x) = (\Delta x)^{\frac{1}{2}} f(k\Delta x).$$

Meanwhile, for the remainder term R^N we obtain

$$\left| \int dx \, R^{(N)}(x) \Phi_{\Delta x}(x - k\Delta x) \right| \leq K \int dx \, |\Phi_{\Delta x}(x - k\Delta x)| \, |x - k\Delta x|^N .$$

Since Φ is assumed to be exponentially decaying, say, $|\Phi(x)| \leq C e^{-\varepsilon |x|}$, it follows that

$$\left| \int dx \, R^{(N)}(x) \Phi_{\Delta x}(x - k\Delta x) \right| \leq K'(\Delta x)^{N - \frac{1}{2}}.$$

Comparing the interpolating approximation of f in $\mathbf{V}_0^{\Delta x}$,

$$\sum_{k=-\infty}^{\infty} (\Delta x)^{\frac{1}{2}} f(k\Delta x) \Phi_{\Delta x}(x - k\Delta x),$$

and the exact projection of f into the subspace $\mathbf{V}_0^{\Delta x}$,

$$f(x) \approx \sum_{k=-\infty}^{\infty} d_k \Phi_{\Delta x}(x - k\Delta x),$$

we see that the first approximation interpolates the values $f(k\Delta x)$, while the error in each coefficient is bounded by

$$\left| (\Delta x)^{\frac{1}{2}} f(k\Delta x) - d_k \right| = O((\Delta x)^{N - \frac{1}{2}}).$$

Because of this estimate, interpolating multiresolutions provide an arguably natural way of computing from sampled data the initial projection needed for Mallat's cascade wavelet transform algorithm [7]. We can interpolate the sampled values while deviating from the exact projection with a controllable error.

10. A technical lemma. In this section we prove a lemma which we shall use to prove exponential decay for the multiresolutions we construct.

LEMMA 10.1. *Suppose that for $\omega \in \mathbf{R}$, $\theta(\omega)$ is 2π-periodic, real-analytic, $|\theta(0)| = 1$, and $|\theta(\omega)| \geq b > 0$.*

Then $\Theta(\zeta) = \prod_{k=1}^{\infty} \theta(2^{-k}\zeta)$ is analytic on some strip $\Gamma = \{\zeta \in \mathbf{C} : |\operatorname{Im} \zeta| < r\}$ containing the real axis, and there exists a constant A such that for all $\omega + i\eta \in \Gamma$, $|\Theta(\omega + i\eta)| \leq A \, |\Theta(\omega)|$.

Proof. Since $\theta(\omega)$ is real-analytic and 2π-periodic, $\theta(\zeta)$ is analytic on some strip $\Omega = \{\zeta \in \mathbf{C} : |\operatorname{Im} \zeta| < R\}$. Because $|\theta(0)| = 1$, if $K \subset\subset \Omega$ is compact, then we can find $C_K > 0$ such that for all $\zeta \in K$,

$$\left| \theta(2^{-k}\zeta) \right| \leq 1 + C_K \, 2^{-k}.$$

This insures that the infinite product defining $\Theta(\zeta)$ converges uniformly on compact subsets of Ω, so Θ is analytic on Ω.

To establish the bound on $\Theta(\zeta)$, we first define for $\omega \in \mathbf{R}$

$$\lambda(\omega) = \frac{\theta'(\omega)}{\theta(\omega)}$$

$$\Lambda(\omega) = \sum_{k=1}^{\infty} 2^{-k} \lambda(2^{-k}\omega).$$

Since $\theta(\omega)$ is 2π-periodic, $\theta'(\omega)$ is bounded and the sum defining $\Lambda(\omega)$ is uniformly convergent.

Since $\lambda(\omega)$ is 2π-periodic, we may write $\lambda(\omega) = \sum_{k=-\infty}^{\infty} \lambda_k e^{ik\omega}$. Because of the analyticity of θ, $\lambda(\zeta)$ is analytic on Ω, so for some $a \in (0,1)$ and $\beta > 1$, independent of k, we have the bound

$$|\lambda_k| \leq \beta e^{-a|k|}.$$

With these definitions,

$$\Theta'(\omega) = \left(\sum_{k=1}^{\infty} 2^{-k} \frac{\theta'(2^{-k}\omega)}{\theta(2^{-k}\omega)} \right) \prod_{\ell=1}^{\infty} \theta(2^{-\ell}\omega) = \Lambda(\omega)\Theta(\omega).$$

By Leibniz' rule, for $n \geq 1$ we have

$$\Theta^{(n)}(\omega) = \sum_{k=0}^{n-1} \binom{n-1}{k} \Lambda^{(k)}(\omega)\Theta^{(n-1-k)}(\omega).$$

We will now bound $\Theta^{(n)}(\omega)$. We claim that for some constant $\gamma > 0$, independent of ω,

$$(29) \qquad \left| \Theta^{(n)}(\omega) \right| \leq n! \frac{\gamma^n}{a^{n+1}} |\Theta(\omega)|.$$

First note that

$$\left| \lambda^{(n)}(\omega) \right| = \left| \sum_{k=-\infty}^{\infty} (ik)^n \lambda_k e^{ik\omega} \right| \leq \sum_{k=-\infty}^{\infty} \beta |k|^n e^{-a|k|}.$$

From the identity

$$\int_0^{\infty} x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

and Stirling's formula we can derive the estimate

$$\left| \lambda^{(n)}(\omega) \right| \leq \gamma \frac{n!}{a^{n+1}}$$

for some $\gamma \geq 1$ which is independent of n. Consequently,

$$\left| \Lambda^{(n)}(\omega) \right| = \left| \sum_{k=1}^{\infty} (2^{-k})^n \lambda^{(n)}(2^{-k}\omega) \right| \leq \sum_{k=1}^{\infty} (2^{-k})^n \gamma \frac{n!}{a^{n+1}} \leq \gamma \frac{n!}{a^{n+1}}.$$

The proof of (29) now proceeds by induction. To begin, observe that

$$|\Theta'(\omega)| = |\Lambda(\omega)| \, |\Theta(\omega)| \leq \frac{\gamma}{a^2} |\Theta(\omega)|.$$

Now suppose that for $k = 0, \ldots, n-1$,

$$\left| \Theta^{(k)}(\omega) \right| \leq k! \frac{\gamma^k}{a^{k+1}} |\Theta(\omega)|.$$

Then

$$\left|\Theta^{(n)}(\omega)\right| \leq \sum_{k=0}^{n-1} \binom{n-1}{k} \left|\Lambda^{(k)}(\omega)\right| \left|\Theta^{(n-1-k)}(\omega)\right|$$

$$\leq \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} \frac{\gamma k!}{a^{k+1}} \frac{\gamma^{n-1-k}(n-1-k)!}{a^{n-k}} |\Theta(\omega)|$$

$$\leq \sum_{k=0}^{n-1} (n-1)! \frac{\gamma^n}{a^{n+1}} |\Theta(\omega)| = n! \frac{\gamma^n}{a^{n+1}} |\Theta(\omega)|,$$

which establishes (29).

Then for any $\omega \in \mathbf{R}$,

$$|\Theta(\zeta)| = \left| \sum_{n=0}^{\infty} \frac{\Theta^{(n)}(\omega)}{n!} (\zeta - \omega)^n \right| \leq \left(\sum_{n=0}^{\infty} \frac{\gamma^n}{a^{n+1}} |\zeta - \omega|^n \right) |\Theta(\omega)|.$$

If $|\zeta - \omega| < a/\gamma$, then the latter infinite sum is convergent and bounded independently of ω. Thus there exists a strip $\Gamma = \{\zeta \in \mathbf{C} : |\operatorname{Im} \zeta| < r\}$ containing the real axis and A such that

$$|\Theta(\omega + i\eta)| \leq A \, |\Theta(\omega)|$$

for all $\omega + i\eta \in \Gamma$. $\square$

11. A method to construct interpolating multiresolutions. In this section we give a method of constructing an interpolating multiresolution $\mathbf{I}$ given another multiresolution $\mathbf{M}$. Recall from Section 4 that for an interpolating multiresolution $\mathbf{I}$ the filter $H_{\mathbf{I}}(\omega) = a(\omega) + ib(\omega)$ is determined once we define its real part $a(\omega)$. The real part $a(\omega)$ must satisfy

$$a(\omega) + a(\omega + \pi) = 1.$$

Happily, examples of such functions $a(\omega)$ are associated with any multiresolution $\mathbf{M}$. Suppose that $H_{\mathbf{M}}(\omega)$ is the associated scale-transition filter; then

$$|H_{\mathbf{M}}(\omega)|^2 + |H_{\mathbf{M}}(\omega + \pi)|^2 = 1.$$

Thus we might take as a candidate for $a(\omega)$ the function $|H_{\mathbf{M}}(\omega)|^2$.

Such a choice of $a(\omega)$ has the following attractive feature. Suppose that ϕ is the scaling function associated with the multiresolution $\mathbf{M}$. Then from (18) we see that

$$(30) \qquad |H_{\mathbf{I}}(\omega)| = a^{\frac{1}{2}}(\omega) = |H_{\mathbf{M}}(\omega)|.$$

Consequently,

$$\left|\hat{\Phi}(\omega)\right| = \prod_{k=1}^{\infty} \left|H_{\mathbf{I}}(2^{-k}\omega)\right| = \prod_{k=1}^{\infty} \left|H_{\mathbf{M}}(2^{-k}\omega)\right| = \left|\hat{\phi}(\omega)\right|.$$

The scaling function of the interpolating multiresolution **I** is as smooth as the scaling function of the multiresolution **M** used to construct it, at least as measured by the decay of the Fourier transform.

In the same sense, the wavelet of the interpolating multiresolution **I** is as smooth as the wavelet of the multiresolution **M** used to construct it. The wavelets are defined by (8) in terms of the filters

$$G_{\mathbf{M}}(\omega) = e^{-i\omega}\overline{H_{\mathbf{M}}(\omega + \pi)}$$
$$G_{\mathbf{I}}(\omega) = e^{-i\omega}\overline{H_{\mathbf{I}}(\omega + \pi)}.$$

From (30) we see that $|G_{\mathbf{I}}(\omega)| = |G_{\mathbf{M}}(\omega)|$, so

$$\left|\hat{\Psi}(\omega)\right| = \left|G_{\mathbf{I}}\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right)\right| = \left|G_{\mathbf{M}}\left(\frac{\omega}{2}\right)\phi\left(\frac{\omega}{2}\right)\right| = \left|\hat{\psi}(\omega)\right|.$$

As we shall see, interpolating multiresolutions so constructed inherit most of the properties of the multiresolution used to construct them. This gives us tremendous latitude in the design of interpolating multiresolutions. We will present two classes of interpolating multiresolutions, one based on the spline multiresolutions of Lemarie and Battle and one based on the compactly supported multiresolutions of Daubechies. In both cases the resulting scaling function Φ and wavelet Ψ are exponentially decaying and have the same smoothness as the multiresolutions used in their construction.

There is one restriction that arises in the construction of an interpolating multiresolution from an arbitrary multiresolution. This restriction arises in order to assure sufficient smoothness of $H_{\mathbf{I}}(\omega)$. We want $H_{\mathbf{I}}(\omega)$ to be C^{∞} in order that the associated scaling function and wavelet are rapidly decaying as in the definition of a regular multiresolution in Section 2. In the examples we will present, $H_{\mathbf{M}}(\omega)$ has the form

$$|H_{\mathbf{M}}(\omega)|^2 = \cos^{2N}\frac{\omega}{2}\,|m(\omega)|^2$$

where $|m(\omega)|^2$ is an even, 2π-periodic function and $|m(\omega)|^2 > 0$ for all $\omega \in \mathbf{R}$. Now construct $a(\omega)$ as described above:

$$a(\omega) = |H_{\mathbf{M}}(\omega)|^2 = \cos^{2N}\frac{\omega}{2}\,|m(\omega)|^2.$$

From (19) and (21) we have

$$b^2(\omega) = a(\omega)\left(1 - a(\omega)\right) = a(\omega)a(\omega + \pi) = \cos^{2N}\frac{\omega}{2}\sin^{2N}\frac{\omega}{2}\,|m(\omega)|^2\,|m(\omega + \pi)|^2.$$

We want to define $b(\omega)$ so that it is continuous and satisfies $b(\omega) + b(\omega + \pi) = 0$. Accordingly, we must define $b(\omega)$ to be

$$b(\omega) = \begin{cases} \sin^N(\omega/2)\cos^N(\omega/2)\,|m(\omega)|\,|m(\omega + \pi)|, & \text{N odd} \\ \sigma(\omega)\sin^N(\omega/2)\cos^N(\omega/2)\,|m(\omega)|\,|m(\omega + \pi)|, & \text{N even,} \end{cases}$$

where $\sigma(\omega)$ is the 2π-periodic function

$$\sigma(\omega) = \begin{cases} 1, & 0 \le \omega < \pi, \\ -1, & \pi \le \omega < 2\pi. \end{cases}$$

If N is even, then $H(\omega)$ cannot be $\mathbf{C}^\infty$. This limits the decay of the associated scaling function Φ and the wavelet Ψ; we cannot expect Φ and Ψ to have more than polynomial decay. For this reason we will limit our attention in the remainder of this paper to the case where N is odd. The constructions we present can be carried out in the case where N is even, however, yielding multiresolutions that are not regular in the sense of Section 2. Numerical calculations indicate that the difference in the asymptotic rate of decay between the cases N even and N odd is not really noticeable.

THEOREM 11.1. *Suppose that $\phi(x)$ is an real-valued, exponentially decaying scaling function such that $(1 + |\omega|)^{1+\alpha} \hat{\phi} \in \mathbf{L}^\infty(\mathbf{R})$ for some $\alpha > 0$, and the associated scale-transition filter has the form $H_\mathbf{M}(\omega) = \cos^N(\omega/2)\, m(\omega)$, where N is odd, $m(\omega)$ is 2π-periodic, $|m(\omega)| \ne 0$ and $|m(\omega)|$ is even.*

Then

$$(31)\ H_\mathbf{I}(\omega) = \frac{|m(\omega)|}{m(\omega)} \left(\cos^N \frac{\omega}{2}\, |m(\omega)| + i \sin^N \frac{\omega}{2}\, |m(\omega + \pi)| \right) H_\mathbf{M}(\omega) \equiv \theta(\omega) H_\mathbf{M}(\omega)$$

defines a regular multiresolution with scaling function Φ and wavelet Ψ with the following properties:

1. *Reality: Φ and Ψ are real-valued.*
2. *Smoothness: Φ and Ψ are as smooth as ϕ and ψ in the sense that for $\omega \in \mathbf{R}$.*

$$\begin{aligned} \left| \hat{\Phi}(\omega) \right| &= \left| \hat{\phi}(\omega) \right| \\ \left| \hat{\Psi}(\omega) \right| &= \left| \hat{\psi}(\omega) \right|. \end{aligned}$$

If m is an integer for which $0 \le m < \alpha$, then Φ and Ψ are $\mathbf{C}^m$ functions.
3. *Interpolation: Φ is continuous, and $\Phi(k) = \delta(k)$.*
4. *Exponential decay: Φ and Ψ are exponentially decaying. If m is an integer for which $0 \le m < \alpha$, then for some $C > 0$ and $\varepsilon > 0$,*

$$\left| \Phi(x) \right|, \left| \Phi'(x) \right|, \dots, \left| \Phi^{(m)}(x) \right|, \left| \Psi(x) \right|, \left| \Psi'(x) \right|, \dots, \left| \Psi^{(m)}(x) \right| \le C e^{-\varepsilon |x|}.$$

5. *Vanishing moments: Φ and Ψ have vanishing moments:*

$$\int dx\, x^k \Phi(x) = 0 \quad \text{for } 1 \le k \le N - 1$$

$$\int dx\, x^k \Psi(x) = 0 \quad \text{for } 0 \le k \le N - 1.$$

Proof. First we must check that $H_\mathbf{I}(\omega)$ defines a multiresolution.

  i. $H_\mathbf{I}(\omega)$ is 2π-periodic and $\mathbf{C}^\infty$ because of the hypotheses on $m(\omega)$.

  ii. Since $H_\mathbf{M}(0) = 1$, $m(0) = 1$, so $H_\mathbf{I}(0) = 1$.

iii. Note that

$$\begin{aligned}
|\theta(\omega)|^2 &= \cos^{2N}\frac{\omega}{2}\,|m(\omega)|^2 + \sin^{2N}\frac{\omega}{2}\,|m(\omega+\pi)|^2 \\
&= |H_{\mathbf{M}}(\omega)|^2 + |H_{\mathbf{M}}(\omega+\pi)|^2 = 1.
\end{aligned}$$

Thus

$$|H_{\mathbf{I}}(\omega)|^2 + |H_{\mathbf{I}}(\omega+\pi)|^2 = |H_{\mathbf{M}}(\omega)|^2 + |H_{\mathbf{M}}(\omega+\pi)|^2 = 1.$$

iv. Since $|H_{\mathbf{I}}(\omega)| = \left|\cos^N\frac{\omega}{2}\right||m(\omega)|$, we see that $|H_{\mathbf{I}}(\omega)| \neq 0$ for $\omega \in [-\pi/2, \pi 2]$. Theorem 3.1 then tells us that $H_{\mathbf{I}}(\omega)$ defines a multiresolution.

Now we proceed to the proof of points 1—4.

1. To prove that Φ and Ψ are real, it suffices to show that $H_{\mathbf{I}}(\omega) = \overline{H_{\mathbf{I}}(-\omega)}$. Since ϕ is real-valued, it follows that $H_{\mathbf{M}}(\omega) = \overline{H_{\mathbf{M}}(-\omega)}$, whence $m(\omega) = \overline{m(-\omega)}$. Then

$$H_{\mathbf{I}}(-\omega) = \frac{|m(-\omega)|}{m(-\omega)}\left(\cos^N\frac{\omega}{2}\,|m(-\omega)| - i\sin^N\frac{\omega}{2}\,|m(-\omega+\pi)|\right)H_{\mathbf{M}}(-\omega).$$

By hypothesis, $|m(\omega)|$ is even and 2π-periodic; hence $|m(-\omega+\pi)| = |m(\omega+\pi)|$. Thus

$$H_{\mathbf{I}}(-\omega) = \overline{H_{\mathbf{I}}(\omega)}.$$

2. We have shown that $|H_{\mathbf{I}}(\omega)| = |H_{\mathbf{M}}(\omega)|$, so part 2 immediately follows.

3. Since $\hat{\phi} \in \mathbf{L}^1$, part 2 tells us that $\hat{\Phi} \in \mathbf{L}^1$, so Φ is continuous. To establish the interpolation identity $\Phi(k) = \delta(k)$ we need to check that $H_{\mathbf{I}}(\omega) + H_{\mathbf{I}}(\omega+\pi) = 1$. We have

$$\begin{aligned}
H_{\mathbf{I}}(\omega) + H_{\mathbf{I}}(\omega+\pi) &= \cos^{2N}\frac{\omega}{2}\,|m(\omega)|^2 + i\sin^N\frac{\omega}{2}\cos^N\frac{\omega}{2}\,|m(\omega)|\,|m(\omega+\pi)| \\
&\quad + \sin^{2N}\frac{\omega}{2}\,|m(\omega+\pi)|^2 - i\sin^N\frac{\omega}{2}\cos^N\frac{\omega}{2}\,|m(\omega)|\,|m(\omega+\pi)| \\
&= |H_{\mathbf{M}}(\omega)|^2 + |H_{\mathbf{M}}(\omega+\pi)|^2 = 1.
\end{aligned}$$

4. We will use the following facts to prove the exponential decay (see [3]).

   a. If $|u(x)| \leq Ce^{-\epsilon|x|}$, then there exists a strip in $\mathbf{C}$ containing the real axis on which $\hat{u}(\zeta)$ is analytic.

   b. If $\hat{u}(\zeta)$ is analytic on some strip Γ in $\mathbf{C}$ containing the real axis and if $\hat{u}(\zeta) = O(|\zeta|^{-(1+\beta)})$, $\beta > 0$ as $\zeta \to \infty$, $\zeta \in \Gamma$, then for some C and $\varepsilon > 0$, $|u(x)| \leq Ce^{-\varepsilon|x|}$.

We have

$$\hat{\Phi}(\omega) = \prod_{k=1}^{\infty}\theta(2^{-k}\omega)\cos^{2N}\left(2^{-k}\frac{\omega}{2}\right)m(2^{-k}\omega) = \left(\frac{\omega}{2}\right)^{-N}\sin^{2N}\frac{\omega}{2}\left(\prod_{k=1}^{\infty}\theta(2^{-k}\omega)m(2^{-k}\omega)\right).$$

Because ϕ is assumed to be exponentially decaying, $\theta(\omega)$ and $m(\omega)$ are real-analytic. Lemma 10.1 then tells us that $\Theta(\zeta) = \prod_{k=1}^{\infty}\theta(2^{-k}\zeta)m(2^{-k}\zeta)$ is analytic in some strip $\Gamma = \{\zeta \in \mathbf{C} : |\operatorname{Im}\zeta| < r\}$, and there exists A such that

$$|\Theta(\omega+i\eta)| \leq A\left|\prod_{k=1}^{\infty}\theta(2^{-k}\zeta)m(2^{-k}\zeta)\right| = A\prod_{k=1}^{\infty}\left|m(2^{-k}\zeta)\right|.$$

Meanwhile, there exists B such that for all $\zeta = \omega + i\eta \in \Gamma$, $|\omega| \geq 1$,

$$\left| \left(\frac{\zeta}{2} \right)^{-N} \sin^{2N} \frac{\zeta}{2} \right| \leq B \left| \left(\frac{\omega}{2} \right)^{-N} \sin^{2N} \frac{\omega}{2} \right|.$$

Consequently, for all $\zeta = \omega + i\eta \in \Gamma$, $|\omega| \geq 1$,

$$\left| \hat{\Phi}(\zeta) \right| \leq AB \left| \left(\frac{\omega}{2} \right)^{-N} \sin^{2N} \frac{\omega}{2} \right| \prod_{k=1}^{\infty} \left| m(2^{-k}\omega) \right| = AB \left| \hat{\phi}(\omega) \right|.$$

Thus we conclude that $\hat{\Phi}(\zeta)$ is analytic on the strip Γ containing the real axis and $\hat{\Phi}(\zeta) = O(|\zeta|^{-(1+\alpha)})$ as $\zeta \to \infty$, so Φ is an exponentially decaying function.

The decay of $\Phi', \ldots, \Phi^{(m)}$ consists of noting that for $0 \leq k \leq m$, $\zeta^k \hat{\Phi}(\zeta)$ is analytic on Γ and is $O(|\zeta|^{-(1+\beta)})$, $\beta > 0$, as $\zeta \to \infty$ for $\zeta \in \Gamma$. The proof of the exponential decay of $\Psi, \Psi', \ldots, \Psi^{(m)}$ follows the same lines.

5. This is simply proposition 9.1. $\square$

From formula (31) we can discern something of the relation between the original scaling function ϕ and the interpolating scaling function Φ constructed from ϕ. We have

$$\hat{\Phi}(\omega) = \left(\prod_{k=1}^{\infty} \theta(2^{-k}\omega) \right) \hat{\phi}(\omega).$$

Writing $\theta(\omega) = \sum_{k=-\infty}^{\infty} \theta_k e^{ik\omega}$, we see that $\theta(2^{-j}\omega)$ is the Fourier transform of the distribution

$$(32) \qquad \Theta_j(x) = \sum_{k=-\infty}^{\infty} \theta_k \, \delta(x - k2^{-k}).$$

Thus Φ is derived from ϕ via an iterated convolution:

$$\Phi(x) = \left(\left(*_{j=1}^{\infty} \Theta_j \right) * \phi \right)(x),$$

which in view of (32) corresponds to an iterated averaging of pointwise function values. This will tend to have a smoothing effect.

12. The interpolating multiresolutions associated with the spline multiresolutions.

In this section we will construct interpolating multiresolutions from the spline multiresolutions of Lemarie and Battle. The spline multiresolution of order m has the following properties (see [4]):

1. The scaling function ϕ and the wavelet ψ are both $\mathbf{C}^{m-2}$ splines. On each interval $[k, k+1]$, $k \in \mathbf{Z}$, ϕ is a polynomial of degree $m-1$, while on each interval $[k/2, (k+1)/2]$, $k \in \mathbf{Z}$, ψ is a polynomial of degree $m-1$.

2. Both ϕ and ψ are exponentially decaying. There exist $C > 0$ and $\varepsilon > 0$ such that

$$|\phi(x)|, |\phi'(x)|, \ldots, \left|\phi^{(m-2)}(x)\right| \leq C e^{-\varepsilon|x|}$$
$$|\psi(x)|, |\psi'(x)|, \ldots, \left|\psi^{(m-2)}(x)\right| \leq C e^{-\varepsilon|x|}.$$

Let H_m denote the scale-transition filter associated with the Lemarie-Battle spline multiresolution of order m. It has the form

$$H_m(\omega) = \sqrt{\frac{\Sigma_{2m}(\omega)}{2^{2m}\Sigma_{2m}(2\omega)}},$$

where

$$\Sigma_{2m}(\omega) = \frac{1}{(2m-1)!}\frac{d^{2m-2}}{d\omega^{2m-2}}\frac{1}{4\sin^2\frac{\omega}{2}} = \frac{P_m(\cos^2\frac{\omega}{2})}{\sin^{2m}\frac{\omega}{2}}.$$

P_m is a polynomial of degree m and $P_m(\cos^2\frac{\omega}{2}) > 0$ for all $\omega \in \mathbf{R}$. The scaling function ϕ and the wavelet ψ associated with the spline multiresolution of order m are given by

$$\begin{aligned}
\hat{\phi}(\omega) &= \omega^{-m}\left(\Sigma_{2m}(\omega)\right)^{-\frac{1}{2}} \\
\hat{\psi}(\omega) &= e^{i\frac{\omega}{2}}\overline{H_m(\omega+\pi)}\hat{\phi}(\frac{\omega}{2}).
\end{aligned}$$

For odd m we can construct an interpolating multiresolution using the method described in Section 11. Set

$$\begin{aligned}
a(\omega) &= |H_m(\omega)|^2 = \cos^{2m}\frac{\omega}{2}\frac{P_m(\cos^2\frac{\omega}{2})}{P_m(\cos^2\omega)}, \\
b(\omega) &= \cos^m\frac{\omega}{2}\sin^m\frac{\omega}{2}\sqrt{\frac{P_m(\cos^2\frac{\omega}{2})P_m(\sin^2\frac{\omega}{2})}{P_m^2(\cos^2\omega)}}.
\end{aligned}$$

Then $H_\mathbf{I}(\omega) = a(\omega) + ib(\omega)$ defines an interpolating multiresolution which inherits the smoothness, decay, and vanishing moment properties of the spline multiresolution. Applying Theorem 11.1, we obtain the following.

PROPOSITION 12.1. *Suppose $H_\mathbf{I}(\omega)$ is defined as above. Then $H_\mathbf{I}(\omega)$ defines a regular multiresolution for which the scaling function Φ and wavelet Ψ have the following properties.*

1. *The scaling function Φ and the wavelet Ψ are both real-valued, $\mathbf{C}^{m-2}$ functions.*
2. *Φ is continuous and $\Phi(k) = \delta(k)$.*
3. *Both Φ and Ψ are exponentially decaying. There exist $C' > 0$ and $\varepsilon' > 0$ such that*

$$\begin{aligned}
|\Phi(x)|, |\Phi'(x)|, \ldots, \left|\Phi^{(m-2)}(x)\right| &\leq C'e^{-\varepsilon'|x|} \\
|\Psi(x)|, |\Psi'(x)|, \ldots, \left|\Psi^{(m-2)}(x)\right| &\leq C'e^{-\varepsilon'|x|}.
\end{aligned}$$

4. *Both Φ and Ψ have vanishing moments:*

$$\begin{aligned}
\int dx\, x^k \Phi(x) &= 0 \qquad \text{for } k = 1, \ldots, m-1, \text{ and} \\
\int dx\, x^k \Psi(x) &= 0 \qquad \text{for } k = 0, \ldots, m-1.
\end{aligned}$$

The scaling function $\phi_m(x)$ of the spline multiresolution of order m approximates a lowpass filter for the frequency interval $[-\pi, \pi]$, which is the Nyquist frequency associated with the sampling interval $\Delta x = 1$. In fact, as $m \to \infty$, $\hat{\phi}_m(\omega) \to \chi_{[-\pi,\pi]}(\omega)$, the characteristic function for $[-\pi, \pi]$ (see [4]). (Of course, the penalty for such spectral localization is that as $m \to \infty$, the function $\phi(x)$ is less localized in space.)

The interpolating scaling function Φ_m has the property that $\left|\hat{\Phi}_m(\omega)\right| = \left|\hat{\phi}_m(\omega)\right|$, so it, too, has most of its frequency content concentrated in the interval $[-\pi, \pi]$. This may make Φ_m an interesting interpolant for sampled functions, where there is an underlying assumption that the data is band-limited.

Figures 1 and 2 show plots of Φ and Ψ for the case $m = 3$.

13. The interpolating multiresolutions associated with the compactly supported multiresolutions.

We can also build interpolating multiresolutions from the multiresolutions of Daubechies [2], for which the associated scaling function ϕ and wavelet ψ are compactly supported. The Daubechies multiresolution of order N has the following properties:

1. ϕ and ψ are compactly supported.
2. ϕ and ψ are smooth, in the sense that there exists $\alpha_N > 0$ such that

$$(1 + |\omega|)^{1+\alpha_N} \left|\hat{\phi}\right| \text{ and } (1 + |\omega|)^{1+\alpha_N} \left|\hat{\psi}\right| \in \mathbf{L}^\infty.$$

The scale-transition filter $H_N(\omega)$ for the Daubechies multiresolution of order N has the form

$$H_N(\omega) = \left[e^{i\frac{\omega}{2}} \cos \frac{\omega}{2}\right]^N Q(e^{i\omega}),$$

where Q is a polynomial such that

$$\left|Q(e^{i\omega})\right|^2 = \sum_{k=0}^{N-1} \binom{N-1+k}{k} \sin^{2k} \frac{\omega}{2} + \left[\sin^{2N} \frac{\omega}{2}\right] R\left(\frac{1}{2} \cos \omega\right),$$

and R is a member of a restricted class of odd polynomials. For our construction we will choose the simplest possibility, $R = 0$.

For odd N we start with $H_N(\omega)$ to define an interpolating multiresolution. Define

$$a(\omega) = |H_N(\omega)|^2 = \cos^{2N} \frac{\omega}{2} \left|Q(e^{i\omega})\right|^2,$$

$$b(\omega) = \cos^N \frac{\omega}{2} \sin^N \frac{\omega}{2} \left|Q(e^{i\omega})\right| \left|Q(e^{i(\omega+\pi)})\right|.$$

Now we can now appeal to Theorem 11.1 to define an interpolating multiresolution from $H_I(\omega) = a(\omega) + ib(\omega)$.

PROPOSITION 13.1. *Suppose $H_I(\omega)$ is defined as above. Then $H_I(\omega)$ defines a regular multiresolution for which the scaling function Φ and wavelet Ψ have the following properties.*

1. *If α_N is the exponent in Property 2 of the compactly supported multiresolution, then*

$$(1 + |\omega|)^{1+\alpha_N} \left| \hat{\Phi} \right| \quad and \quad (1 + |\omega|)^{1+\alpha_N} \left| \hat{\Psi} \right| \in \mathbf{L}^\infty.$$

2. *Φ is continuous and $\Phi(k) = \delta(k)$.*

3. *Φ and Ψ both have vanishing moments:*

$$\int dx\, x^k \Phi(x) \;=\; 0 \; for \; k = 1, \ldots, N-1, \; and$$

$$\int dx\, x^k \Psi(x) \;=\; 0 \; for \; k = 0, \ldots, N-1.$$

4. *If m is an integer for which $0 \le m < \alpha_N$, then Φ and Ψ are both $\mathbf{C}^m$. Moreover, Φ, Ψ and their derivatives are all exponentially decaying. There exist $C > 0$ and $\varepsilon > 0$ such that*

$$|\Phi(x)|, |\Phi'(x)|, \ldots, \left| \Phi^{(m)}(x) \right| \;\le\; C e^{-\varepsilon|x|}$$

$$|\Psi(x)|, |\Psi'(x)|, \ldots, \left| \Psi^{(m)}(x) \right| \;\le\; C e^{-\varepsilon|x|}.$$

Plots of Φ and Ψ for the case $N = 3$ are given in Figures 3 and 4. To the naked eye, these appear to be a little smoother than the plots of ϕ and ψ for the compactly supported multiresolution of order 3 given in [2]. This is what we might expect in view of the averaging effect discussed at the end of Section 11.

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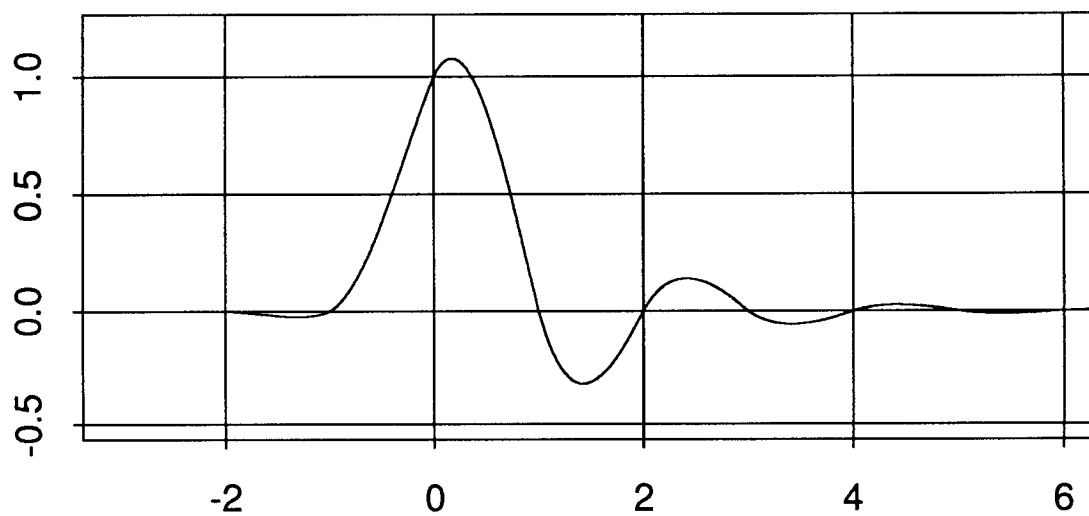[9] G. PÓLYA AND G. SZEGÖ, *Problems and Theorems in Analysis*, vol. 2, Springer-Verlag, 1976.

Figure 1: Interpolating scaling function
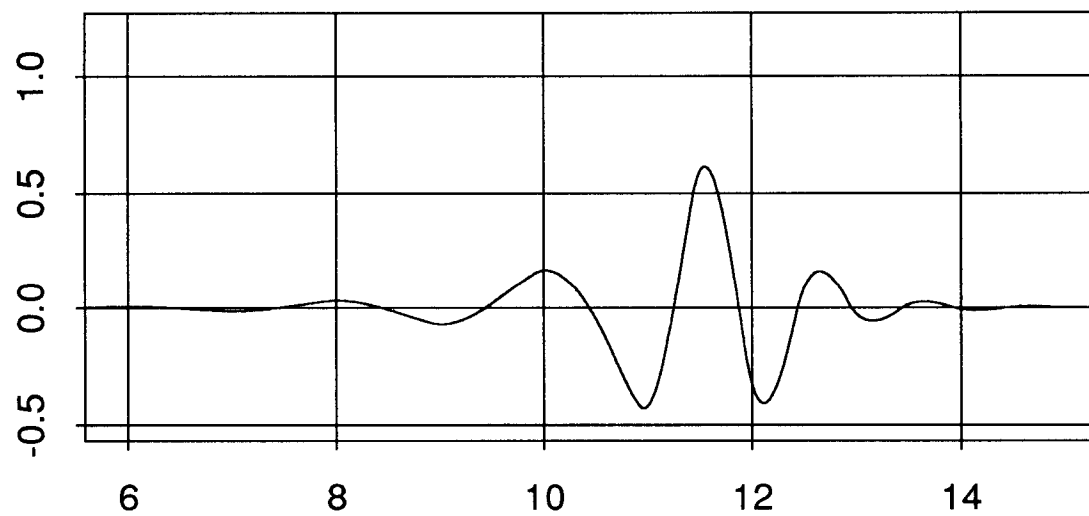built from spline multiresolution, m = 3

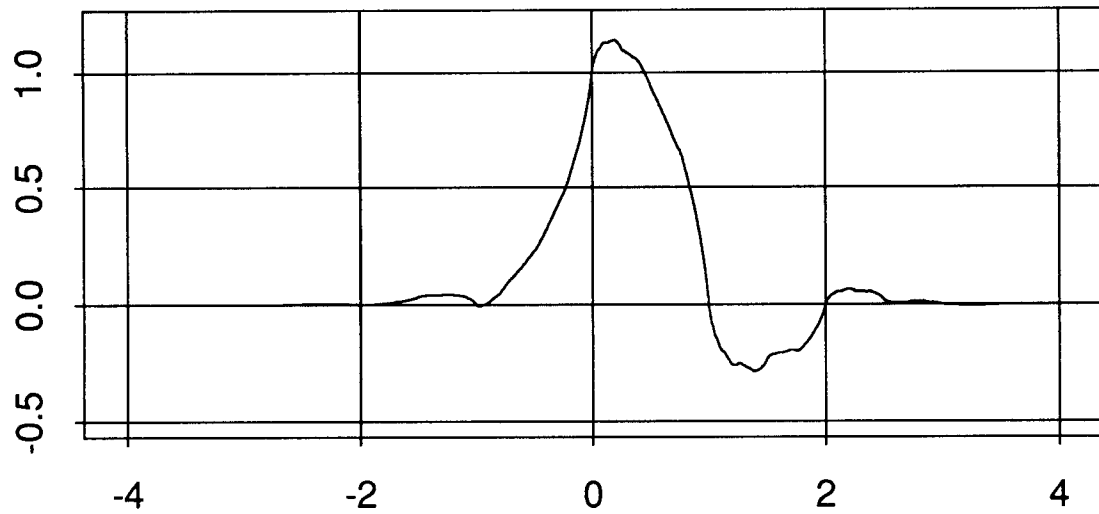Figure 2: Associated wavelet built
from spline multiresolution, m = 3

Figure 3: Interpolating scaling function built
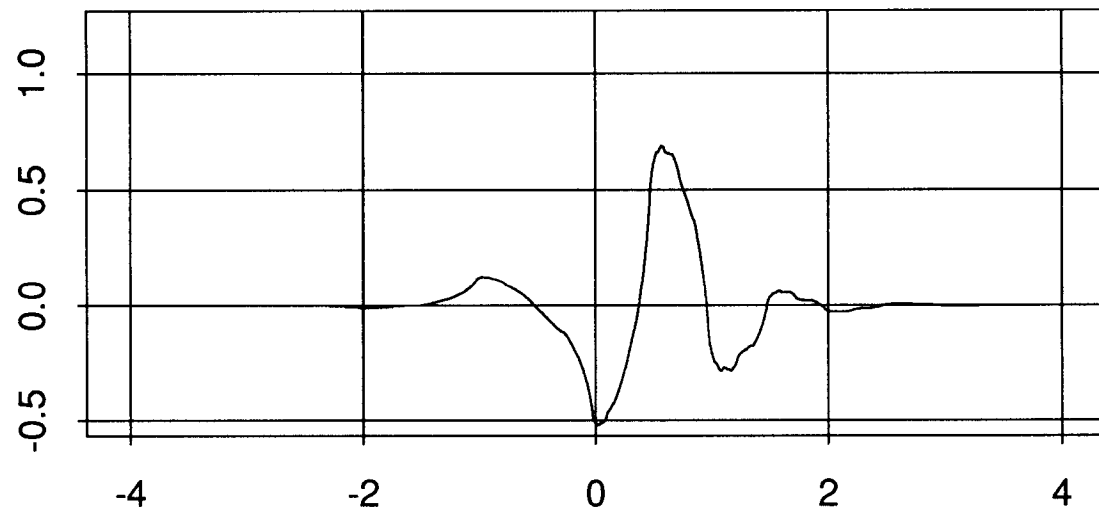from Daubechies multiresolution, N = 3

Figure 4: Associated wavelet built from
Daubechies multiresolution, N = 3